# PLASTICITY CONDITIONS FOR THIN SHELLS

## (OB USLOVII PLASTICHNOSTI DLIA TONKOSTENNYKH OBOLOCHEK)

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Plasticity conditions (finite relation) for thin shells using the Kirchhoff-Love hypothesis and von Mises' plasticity criterion were considered in [1]. Plasticity conditions for the axisymmetrically-loaded cylindrical shell corresponding to the maximum shear-stress criterion were given in [2,3] and the same for the shells of revolution in [4]. In special cases approximate plasticity conditions are usually introduced which are obtained from approximating the above-mentioned exact conditions [2,3,5-7], or from other considerations, [8,9].

Using the extremum principles obtained for the three-dimensional rigid-plastic continuum, one can derive a sufficiently simple approximate condition for a general case. It appears thereby that such a general approximate condition contains the approximate plasticity conditions introduced by the above-mentioned authors.

**1.** We shall use the following relationships expressing the stresses and moments in a shell:

$$
\begin{aligned}
T_1 &= \int_{-1/2h}^{1/2h} \sigma_1 dz, \qquad T_2 = \int_{-1/2h}^{1/2h} \sigma_2 dz, \qquad T_{12} = \int_{-1/2h}^{1/2h} \sigma_{12} dz \\
M_1 &= \int_{-1/2h}^{1/2h} \sigma_1 z dz, \qquad M_2 = \int_{-1/2h}^{1/2h} \sigma_2 z dz, \qquad M_{12} = \int_{-1/2h}^{1/2h} \sigma_{12} z dz
\end{aligned}
\tag{1.1}
$$

Clearly (1.1) is applicable if the ratio of the thickness of the shell $h$ to the characteristic radius of curvature is small in comparison with unity. In the future we shall limit ourselves to such cases only. Let us consider now an element of a plate, whose sides are of unit length, loaded along its edges as shown in Fig. 1. As a statically admissible stress field for a given element we take

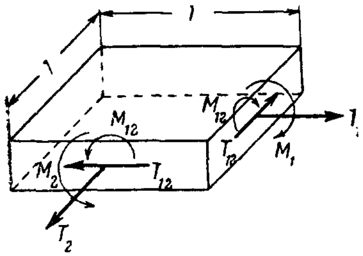


FIG. 1.

$$\sigma_1 = \frac{T_1}{h} + \frac{4M_1}{h^2}\,\text{sign}\, z, \qquad \sigma_{12} = \frac{T_{12}}{h} + \frac{4M_{12}}{h^2}\,\text{sign}\, z$$

$$\sigma_2 = \frac{T_2}{h} + \frac{4M_2}{h^2}\,\text{sign}\, z, \qquad \sigma_{13} = \sigma_{23} = \sigma_3 = 0 \qquad (1.2)$$

(The $z$-coordinate is counted from the middle surface along a local normal.)

Introducing these magnitudes into von Mises' plasticity condition we obtain

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 + 6\,(\sigma_{12}^{\;2} + \sigma_{13}^{\;2} + \sigma_{23}^{\;2}) = 2\sigma_S^2 \qquad (1.2a)$$

We finally deduce relationships resulting from the lower bound estimate

$$P_t^2 + P_m^2 + 2\,|P_{tm}| = 1 \qquad (1.3)$$

where $P_t^{\,2}$, $P_m^{\,2}$, $P_{tm}$ are quadratic and bilinear forms [ 1 ]:

$$\begin{aligned} P_t^2 &= t_1^2 - t_1 t_2 + t_2^2 + 3t_{12}^2 \\ P_m^2 &= m_1^2 - m_1 m_2 + m_2^2 + 3m_{12}^2 \\ 2P_{tm} &= 2t_1 m_1 + 2t_2 m_2 - t_1 m_2 - t_2 m_1 + 6t_{12} m_{12} \end{aligned} \qquad (1.4)$$

Thereby

$$t_1 = \frac{T_1}{T_s}, \qquad m_1 = \frac{M_1}{M_s} \text{ etc. }, \qquad T_s = h\sigma_s, \qquad M_s = \frac{h^2 \sigma_s}{4} \qquad (1.5)$$

From the inequality $P_t P_m > |P_{tm}|$, indicated in [ 1 ], it follows that the replacement of (1.3) by the relation

$$P_t^2 + P_m^2 + 2P_t P_m = 1 \qquad (1.6)$$

leads to a lower bound of the carrying capacity.

In the $P_t P_m$ plane (Fig. 2) the condition (1.6) corresponds to a straight line *AB*.

**2.** We use now the well-known minimum properties [ 10 ] of the functional

$$J = \tau_s \int\limits_V H dv - \int\limits_S (X_n u + Y_n v + Z_n w)\, dS \qquad (2.1)$$

where

$$H = \sqrt{\frac{2}{3}}\left[(\xi_1-\xi_2)^2+(\xi_2-\xi_3)^2+(\xi_3-\xi_1)^2+\frac{3}{2}(\eta_{12}^2+\eta_{23}^2+\eta_{31}^2)\right]^{\frac{1}{2}} \tag{2.2}$$

(the second term in (2.1) expresses the intensity of the given external forces). Consider now the following kinematically admissible field of velocity, which corresponds to a uniformly deformed state:

$$\xi_1 = e_1, \quad \xi_2 = e_2, \quad \xi_3 = -(\xi_1+\xi_2), \quad \eta_{12}=\gamma, \quad \eta_{13}=\eta_{23}=0 \tag{2.3}$$

Thus, (2.1) becomes

$$J = \frac{2}{\sqrt{3}}\sqrt{e_1^2+e_1e_2+e_2^2+\frac{1}{4}\gamma^2} - (t_1e_1+t_2e_2+t_{12}\gamma) \tag{2.4}$$

Determining the parameters $e_1$, $e_2$ and $\gamma$ from

$$\frac{\partial J}{\partial e_1}=0, \qquad \frac{\partial J}{\partial e_2}=0, \qquad \frac{\partial J}{\partial \gamma}=0$$

we find

$$t_1 = \frac{1}{\sqrt{3}}\frac{2e_1+e_2}{\sqrt{e_1^2+e_1e_2+e_2^2+\frac{1}{4}\gamma^2}}$$

$$t_2 = \frac{1}{\sqrt{3}}\frac{2e_2+e_1}{\sqrt{e_1^2+e_1e_2+e_2^2+\frac{1}{4}\gamma^2}} \tag{2.5}$$

$$t_{12} = \frac{1}{2\sqrt{3}}\frac{\gamma}{\sqrt{e_1^2+e_1e_2+e_2^2+\frac{1}{4}\gamma^2}}$$

Elimination of the ratios $e_1/\gamma$, $e_2/\gamma$ from (2.5) leads to the plasticity condition

$$t_1^2 - t_1t_2 + t_2^2 + 3t_{12}^2 = P_t = 1 \tag{2.6}$$

If, however, instead of (2.3) the strain rates are given in the form

$$\xi_1 = z\varkappa_1, \quad \xi_2 = z\varkappa_2, \quad \xi_3 = -(\xi_1+\xi_2), \quad \eta_{12} = z\omega, \quad \eta_{13}=\eta_{23}=0 \tag{2.7}$$

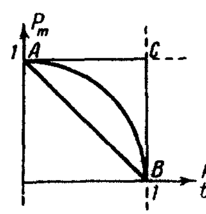


FIG. 2.

we obtain the plasticity condition

$$m_1^2 - m_1m_2 + m_2^2 + 3m_{12}^2 \equiv P_m^2 = 1$$

Thus, the plasticity condition based on the upper-bound approximation* is expressed in the $P_t$ - $P_m$ plane (Fig. 2) by the sides of the square *ACB*. It is then natural to take as an approximate plasticity condition some

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* Notice that the given kinematically-admissible fields in the form of a sum of (2.3) and (2.7) would lead to the plasticity condition [1].

curve located between the lower bound *AB* and the upper bound *ACB*. The simplest assumption then will be

$$P_t^2 + P_m^2 = 1 \tag{2.9}$$

which corresponds to the circular arc *AB*.

**3.** We shall consider the condition (2.9) in greater detail. For the membrane (momentless) and for the pure bending states of stress, the upper and lower-bound evaluations found in Sections 1 and 2, coincide (points *A*, *B* in Fig. 2.) In these cases the solution (2.9) is identical with the plasticity condition presented in [1].

For the axisymmetrically loaded cylindrical shell (in the absence of the axial force) we have

$$t_1 = t_{12} = 0, \qquad m_2 = \frac{1}{2} m_1, \qquad m_{12} = 0$$

The expression (2.9) has the form

$$t_1^2 + \frac{3}{4} m_1^2 = 1$$

which again coincides with the limit relationship used for the solution of this problem in [1]. Let, for the axisymmetrical deformations of the shells of revolution

$$P_t^2 = t^2 = t_2^2 - t_1 t_2 + t_2^2, \qquad P_m^2 = m^2 = m_1^2 - m_1 m_2 + m_2^2 \tag{3.1}$$

The plasticity condition (2.9) is

$$t^2 + m^2 = 1 \tag{3.2}$$

We use a traditional method of piecewise approximation

$$t \approx \tau, \qquad m \approx \mu \tag{3.3}$$

where

$$\tau = \max\{|t_1|, \ |t_2|, \ |t_1 - t_2|\}, \qquad \mu = \max\{|m_1|, |m_2|, |m_1 - m_2|\} \tag{3.4}$$

Here, instead of (3.2), we obtain

$$\tau^2 + \mu^2 = 1 \tag{3.5}$$

The next step consists of replacing a circular arc (3.5) by a circumscribed or inscribed polygon or a square.

$$|\tau| \leqslant 1, \qquad |\mu| \leqslant 1 \tag{3.6}$$

The obtained piecewise linear plasticity condition coincides with the condition developed in [7, 11], where it was assumed in addition, that $m_2 = 0$. The relationships (3.4) and (3.6) include the plasticity square for the axisymmetrically and axially loaded cylindrical shell. The axial force was introduced in [3] and in a series of other works.

In conclusion, we note that the energy theorems obtained in [9] for a somewhat different plasticity condition can be easily extended to the plasticity condition expressed by (3.2).

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